L10 Jan 27 Complete

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In Mathematical Analysis, after learning
convergence of sequence
uniqueness of limit

Qu. What next ?

Recall The definition of a Cauchy sequence

Definition A sequence $(x_n)_{n=1}^{\infty}$ in (X,d) is Cauchy if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall m,n \geq N \quad d(x_m, x_n) < \varepsilon$

Qu Can it be defined on a topological space?

This topic is only for metric space

Definition. A metric space (X,d) is complete if every Cauchy sequence converges in X.

Examples

① $\mathbb{R}, \mathbb{R}^n$ are complete

② Any $[a,b]$ is complete

③ $\mathbb{R}^2 \setminus \{(0,0)\}$ is not complete

Fact Every convergent sequence is Cauchy

Key idea Use Δ-inequality near the limit

Philosophy

Example ③ above: take away a limit from $\mathbb{R}^2$

From $\mathbb{Q}$ to $\mathbb{R}$: Insert all limits for Cauchy Sequences

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Qu. Why is Cauchy sequence important?

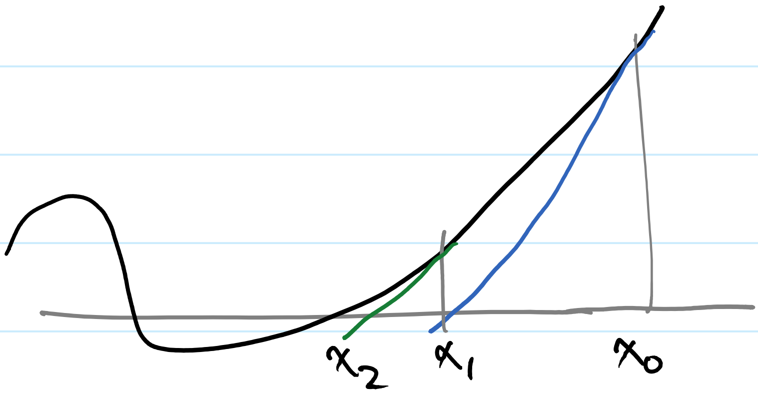
Example Newton's method

To find a solution for $f(x)=0$

Pick $x_0 \in \mathbb{R}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_n \longrightarrow$ a root



Qu. How do we know (x_n) converges?

With reasonable condition on f', we prove (x_n) is a Cauchy sequence (contraction)

Philosophy

Cauchy sequence (in a complete metric space) may help us to show something exists!

Qu Let (X,d) be a complete metric space. $Y \subset X$, using the same metric d. Is (Y,d) complete?

Obviously, $\mathbb{R}^2 \setminus \{(0,0)\}$ is an answer

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Proposition. Let (X,d) be complete and $Y \subset X$

(Y,d) is complete $\iff$ $\underbrace{Y \text{ is closed in } X}_{Y \supset \overline{Y} \ni \text{cluster pts.}}$

"$\Leftarrow$" Let (y_n) be Cauchy in Y

$\therefore$ it is Cauchy in X (same metric)

$\therefore$ $y_n \longrightarrow x \in X$ (X is complete)

$\therefore$ $x \in \overline{Y}$ (Exercise)

$\overline{Y} = Y$ (Y is closed)

"$\Rightarrow$" Let $x \in \overline{Y}$

$\therefore \exists\, y_n \in Y,\ y_n \longrightarrow x$ (metric space)

$\therefore$ (y_n) is Cauchy in X and so in Y

$\therefore$ $y_n \longrightarrow y \in Y \subset X$

By uniqueness, $x = y \in Y$. $\square$

Qu. Think of a theorem in Mathematical Analysis that Cauchy sequence is used in the proof.

Examples.

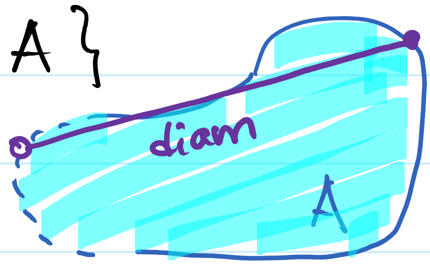
Nested Interval Theorem

Continuous function on $[a,b]$ has max & min

Note. This one has another proof

Diameter On a metric space (X,d), $A \subset X$

$\mathrm{diam}(A) = \{ d(a_1,a_2) : a_1, a_2 \in A \}$



Cantor Intersection Theorem Let (X,d) be a $\boxed{\text{complete}}$ metric space; $\phi \neq F_n \subset X$;

* each F_n is closed
* $F_{n+1} \subset F_n$
* $\mathrm{diam}(F_n) \to 0$ as $n \to \infty$

Then $\bigcap_{n=1}^{\infty} F_n$ is a singleton

Existence, i.e., nonempty

Uniqueness

Qu How to start the proof?

Obviously, create a Cauchy sequence (x_n)

By completeness of X, $x_n \longrightarrow$ desired point

* Pick $x_n \in F_n$ for each $n \in \mathbb{N}$ (What else!)
* $d(x_{n+p}, x_n) \leq \mathrm{diam}(F_n) \longrightarrow 0$, $\therefore$ Cauchy

  By completeness of X, $x_n \longrightarrow$ some $x \in X$

* **Why** $x \in F_n \ \forall n$?

  x is the limit of $(x_n, x_{n+1}, x_{n+2}, \cdots)$ in F_n

  $\therefore x \in \overline{F_n} = F_n$ (closed)

* $\mathrm{diam}(F_n) \to 0$ again $\Rightarrow$ unique.

Contraction Mapping $f:(X,d_X)\to(X,d_X)$ is a contraction mapping if $\exists$ constant $0<\alpha<1$ such that for all $x_1,x_2\in X$

$$d_X(f(x_1),f(x_2)) < \alpha\, d_X(x_1,x_2)$$

Note. Similar to Lipschitz (see later)

Banach Fixed Point Theorem

A contraction mapping on a complete metric space has a fixed point.

That is, $\exists\ x_0\in X$ such that $f(x_0)=x_0$.

Qu. How to construct the fixed point?

No other method, try our luck!

Any $x_1\in X$, then

$$x_2=f(x_1),\ \dots,\ x_{n+1}=f(x_n)$$

Need to show (x_n) is a Cauchy Sequence

Suppose it is, then by completeness of X,

$$x_n\longrightarrow \text{some } x_0\in X$$

Then

$$\begin{array}{ccc} x_{n+1} & = & f(x_n) \\ \downarrow & & \downarrow\ \text{why?} \\ x_0 & = & f(x_0) \end{array}$$

Metric space is Hausdorff

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Why is $(x_n)_{n=1}^{\infty}$ Cauchy?

$$d(x_{n+p}, x_n) = d(f(x_{n+p-1}), f(x_{n-1}))$$
$$< \alpha\, d(x_{n+p-1}, x_{n-1})$$
$$\vdots$$
$$< \alpha^{n-1} \underbrace{d(x_{p+1}, x_1)}_{\text{does not work depends on } p!}$$

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1}))$$
$$< \alpha\, d(x_n, x_{n-1})$$
$$\vdots$$
$$< \alpha^{n-1} d(x_2, x_1) \quad \text{for } n \geqslant 2$$

$$d(x_{n+p}, x_n) \leqslant d(x_{n+p}, x_{n+p-1}) + \cdots + d(x_{n+1}, x_n)$$
$$< \underbrace{(\alpha^{n+p-1} \cdots + \alpha^{n-1})}_{< \varepsilon \text{ if } n \text{ is large}} d(x_2, x_1)$$

Why is f continuous?

If $g : (X, d_X) \longrightarrow (Y, d_Y)$ satisfies
$$d_Y(g(x_1), g(x_2)) \leqslant C\, d_X(x_1, x_2)$$
then g is continuous

(a) $\forall\, \varepsilon > 0$ take $\delta = \varepsilon / C$

(b) $g\left(B_X\left(x, \frac{\varepsilon}{C}\right)\right) \subset B_Y(f(x), \varepsilon)$

Known Before If $f, g : X \longrightarrow$ Hausdorff are continuous; $A \subset X$ with $\bar{A} = X$ such that $f|_A \equiv g|_A$. then $f \equiv g$ on X

This is a uniqueness statement, need to known that f, g are already continuous on X.

Theorem Let (X, d_X) and (Y, d_Y) be metric spaces and Y be $\boxed{\text{complete.}}$ If $\bar{A} = X$ $f : A \longrightarrow Y$ is uniformly continuous then $\exists$ unique continuous $\hat{f} : X \longrightarrow Y$ such that $\hat{f}|_A = f$

* Both X, Y have to be metric
* f only initially defined on A, not X
* Need a "stronger" continuity (only for metric)
* $\hat{f}$ is also "stronger" continuous
* Uniqueness comes from previous theorem

Uniform Continuous $f : (X, d_X) \longrightarrow (Y, d_Y)$ is uniformly continuous if $\forall\ \varepsilon > 0$ $\exists\ \delta > 0$ (only depends on ε) such that

$$\underbrace{\text{if } d(x_1, x_2) < \delta \text{ then } d(f(x_1), f(x_2)) < \varepsilon}$$

$$\forall\ x \in X \quad f(B_X(x, \delta)) \subset B_Y(f(x), \varepsilon)$$

Try to define $\hat{f}(x)$ for $x \in X = \bar{A}$

For each $x \in X$, $\exists$ sequence in A,

call it $a_n^x \longrightarrow x$ as $n \to \infty$

If $x \in A$, choose $a_n^x = x \ \forall n$

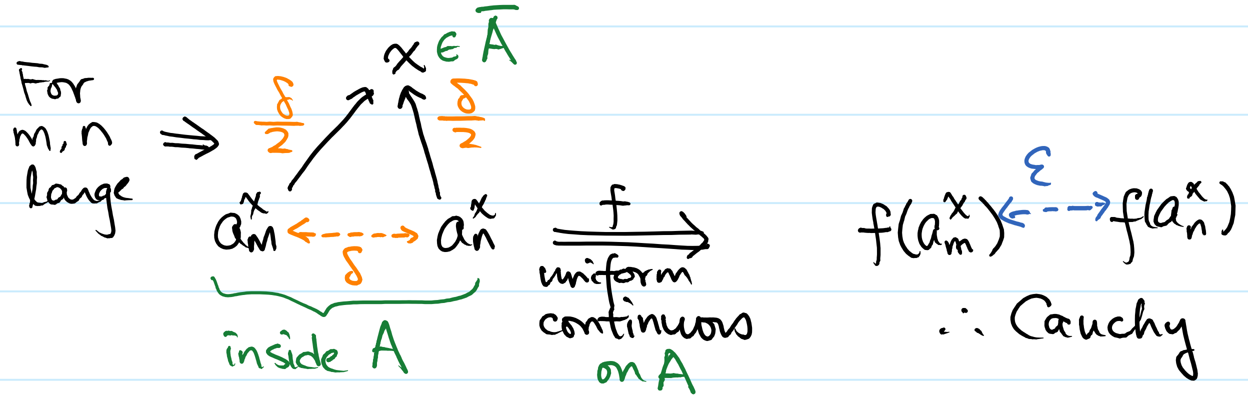
Temporarily, $\hat{f}(x)$ depends on the choice of a_n^x

Then $f(a_n^x)$ is a sequence in Y

Hope. It is Cauchy, $\therefore$ call its limit $\hat{f}(x)$

Take any $\varepsilon > 0$, want to find $N \in \mathbb{N}$ such that

$\forall \ m, n \geq N \quad d_Y(f(a_m^x), f(a_n^x)) < \varepsilon$



For the given $\varepsilon > 0$ above, by unif. continuity of f

$\exists \ \delta > 0$ such that if $a, a' \in A$ with

$d_X(a, a') < \delta \quad$ then $d_Y(f(a), f(a')) < \varepsilon$

For such $\delta > 0$, as (a_n^x) converges to x,

$\exists \ N \in \mathbb{N}$ such that if $m, n \geq N$

$$d_X(a_m^x, a_n^x) \leq d_X(a_m^x, x) + d_X(a_n^x, x) < \delta$$

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Thus, we have $N \in \mathbb{N}$, if $m, n \geq N$

$$d_Y\left(f(a_m^x), f(a_n^x)\right) < \varepsilon$$

The sequence $\left(f(a_n^x)\right)_{n=1}^{\infty}$ in Y is Cauchy and has a limit, to be defined as $\hat{f}(x)$

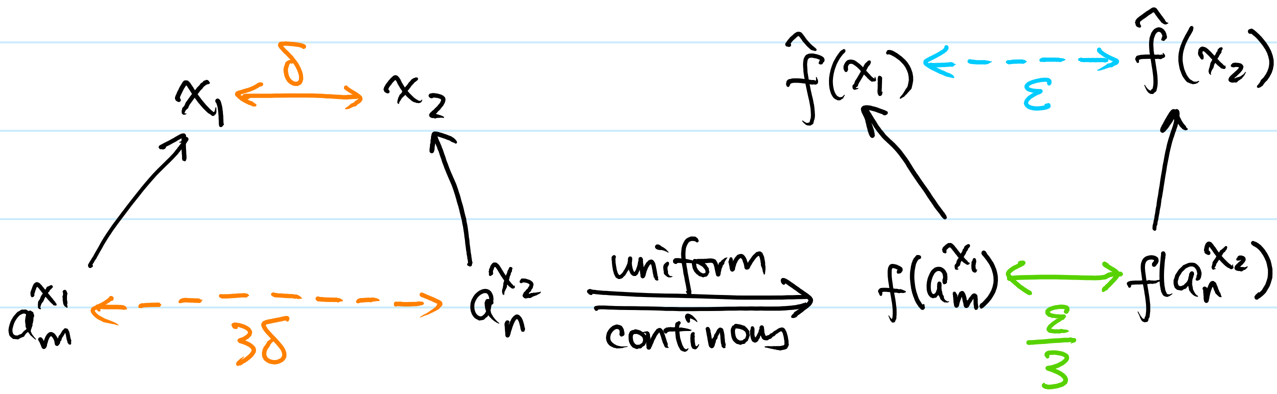
Note that $\hat{f}|_A \equiv f$ by choice of sequence.

If $\hat{f}$ is continuous on X, then by previous result, it is unique, and so indep. of choices of $(a_n^x)_{n=1}^{\infty}$

Continuity of $\hat{f}$ (uniformly)

Want: $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ such that

if $d_X(x_1, x_2) < \delta$ then $d_Y(\hat{f}(x_1), \hat{f}(x_2)) < \varepsilon$



Exercise. Write down the ε-δ argument using the idea of the diagram.